

Vacuum spacetimes with a spacelike, hypersurface-orthogonal Killing vector: reduced equations in a canonical frame

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Abstract.

The Newman-Penrose equations for spacetimes having one spacelike Killing vector are reduced – in a geometrically defined “canonical frame” – to a minimal set, and its differential structure is studied. Expressions for the frame vectors in an arbitrary coordinate basis are given, and coordinate-independent choices of the metric functions are suggested which make the components of the Ricci tensor in the direction of the Killing vector vanish.

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1. Introduction

Interest in the vacuum Einstein equations with a non-null Killing vector (KV) has been rekindled recently by the discovery of Fayos and Sopuerta [1] that the Weyl scalars in the Newman-Penrose (NP) formalism can be expressed algebraically in terms of the spin coefficients and the norm of the Killing vector. Thus, Steele [2] has extended the results of Fayos and Sopuerta to the case of a proper homothetic vector, while Ludwig [3], using the GHP formalism, further extended these results to the non-vacuum case and showed how the formalism can be used to determine all conformal Killing vectors of a particular non-vacuum metric.

The formalisms developed in these papers are completely general. The components of the Killing vector and of the “Papapetrou field” [4] – the exterior derivative of the Killing 1-form – are written down without making any gauge choices (except in [3], where a “preferred tetrad” similar to the one defined here, is used), and the integrability conditions relating these quantities are found. As a result, these formalisms involve a number of variables that can be chosen arbitrarily (to fix the frame relative to the Killing vector), resulting in a redundant set of equations. Moreover, due to the complexity of these equations, the implications of the vacuum Bianchi identities (when the Weyl scalars are substituted for in terms of the spin coefficients) are not considered in any of the works mentioned above. Thus, despite the great number of equations contained in these papers, the entire system of equations to be solved has yet to be written down.

In this paper we examine the particular case of a vacuum spacetime with a spacelike, hypersurface-orthogonal Killing vector, whose norm has a spacelike gradient. These assumptions hold true, at least near infinity, for asymptotically flat axisymmetric spacetimes. Thus, they are appropriate for studying the simplest physical problem where the full dynamical content of General Relativity can be manifested: the coalescence of two Schwarzschild black holes falling toward one another along the line joining their centers. Our choice of notation will refer to this axisymmetric problem, even though the equations will be valid in any spacetime having the assumed properties.

Our main result is that, by making a natural choice of frame based on the assumed properties of the KV, we are able to show that the entire set of Ricci and Bianchi equations in the NP formalism reduce to a set of 17 real equations (11 for vacuum). We also recover, in a coordinate-free way, Waylen's [5] result that, when the "main" equations are satisfied, the contracted Bianchi identities can be formally integrated in terms of a function satisfying the wave equation.

In section 2 we define the canonical frame (up to a boost) and deduce relationships among the spin coefficients that follow from our choice of frame, while in section 3 we obtain the reduced set of equations and elucidate their structure. In section 4 we give general coordinate expressions for the frame components which make the Ricci tensor have vanishing components in the direction of the KV. Finally, in section 5 we discuss ways of choosing the remaining frame and coordinate freedom, as well as additional conditions that can be imposed in order to seek solutions satisfying extra physical or mathematical requirements.

2. The Canonical Frame and the Papapetrou Field

The NP equations are invariant under arbitrary frame rotations (arbitrary Lorentz transformations). We can use this six-real-parameter freedom to fix the frame relative to the geometric structures assumed for the spacetime.

First, we will use three parameters to rotate our frame so that the space-like Killing vector ξ points in the direction of $(\mathbf{m} - \overline{\mathbf{m}})$. (We use the standard NP notation [6]: the complex null-tetrad basis $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \overline{\mathbf{m}}\}$ is normalized to $\mathbf{l} \cdot \mathbf{n} = -\mathbf{m} \cdot \overline{\mathbf{m}} = 1$.) If the (closed) Killing trajectories are parametrized by the parameter φ , then the assumption that the KV is hypersurface orthogonal implies that we can write

$$\xi^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial \varphi} = \mathcal{R} \frac{(m^a - \overline{m}^a)}{i\sqrt{2}} \frac{\partial}{\partial x^a}, \text{ and} \quad (1)$$

$$\xi_a \mathbf{d}x^a = -\mathcal{R}^2 \mathbf{d}\varphi = \mathcal{R} \frac{(m_a - \overline{m}_a)}{i\sqrt{2}} \mathbf{d}x^a, \quad (2)$$

where \mathcal{R} is a scalar function giving the norm of the KV:

$$\xi^a \xi_a = -\mathcal{R}^2. \quad (3)$$

The 3-surfaces orthogonal to the Killing trajectories are spanned by the vectors $\mathbf{l}, \mathbf{n}, (\mathbf{m} + \overline{\mathbf{m}})$. In the following, we will restrict the symbols $\{\mathbf{l}, \mathbf{n}, \mathbf{m}, \overline{\mathbf{m}}\}$ to denote

the corresponding co-vectors (differential forms) while the vectors (differential operators) will be denoted by the standard symbols $\{D, \Delta, \delta, \bar{\delta}\}$. Thus the exterior derivative of an arbitrary function f will be written

$$\mathbf{d}f = \mathbf{l} \Delta f + \mathbf{n} Df - \mathbf{m} \bar{\delta} f - \bar{\mathbf{m}} \delta f. \quad (4)$$

The choice of frame $\mathbf{m} - \bar{\mathbf{m}} \sim$ to a hypersurface orthogonal Killing vector drastically simplifies the NP equations: all spin coefficients become real (see the Appendix), while $\bar{\delta} f = \delta f$ on any function f of the 3 essential coordinates. Thus we can drop the bar from $\bar{\delta}$ (and, of course, from the spin coefficients) in all equations.

Next, we note that the gradient of the norm of the KV will have components in the $\{\mathbf{l}, \mathbf{n}, \mathbf{m} + \bar{\mathbf{m}}\}$ directions only and, by assumption, will be spacelike. We can, therefore, use two degrees of freedom to further restrict our frame by requiring that

$$\mathbf{d}\mathcal{R} \sim \mathbf{m} + \bar{\mathbf{m}} \quad \Longleftrightarrow \quad D\mathcal{R} = 0 = \Delta\mathcal{R}. \quad (5)$$

The frame is now completely determined up to a boost in the $\{\mathbf{l}, \mathbf{n}\}$ plane parametrized by an arbitrary scalar function A :

$$\mathbf{l} \rightarrow A \mathbf{l}, \quad \mathbf{n} \rightarrow \frac{1}{A} \mathbf{n}. \quad (6)$$

There does not seem to be a “natural” choice for eliminating this freedom, based on the assumed properties of the Killing vector. We will discuss possible choices of this last gauge degree of freedom in section 5.

Now, evaluating[‡] $\mathbf{d}[(\mathbf{m} - \bar{\mathbf{m}})/\mathcal{R}] = 0$, which follows from equation (2) with all spin coefficients real and using equation (5), we find

$$\lambda = \mu, \quad \sigma = \rho, \quad \delta \mathcal{R} = -(\alpha - \beta) \mathcal{R}, \quad (7)$$

which determines the proportionality factor in (5) so that the complex co-vector \mathbf{m} takes the form

$$\mathbf{m} = \frac{\mathbf{d}\mathcal{R}}{2(\alpha - \beta)\mathcal{R}} - \frac{\mathbf{i}}{\sqrt{2}} \mathcal{R} \mathbf{d}\varphi. \quad (8)$$

Next, evaluating the condition $\mathbf{d}\mathbf{d}\mathcal{R} = 0$ using equations (5) and (7), we obtain

$$D(\alpha - \beta) = 2\rho(\alpha - \beta) \quad (9)$$

$$\Delta(\alpha - \beta) = -2\mu(\alpha - \beta) \quad (10)$$

$$\pi = -\tau. \quad (11)$$

At this point it is convenient to introduce another scalar function, \mathcal{Q} , by the equation

$$(\alpha - \beta)\mathcal{R} = -\mathcal{Q}/\sqrt{2}, \quad \text{so that } \mathcal{R}^a \mathcal{R}_{,a} = -\mathcal{Q}^2. \quad (12)$$

The function \mathcal{Q} takes the value of unity at infinity (asymptotic flatness), and on the axis ($\mathcal{R} = 0$) when the coordinate φ has the standard periodicity of 2π (“regularity condition” – see [7]). In terms of \mathcal{Q} , equations (9), (10) take the form

$$D\mathcal{Q} = 2\rho\mathcal{Q}, \quad \Delta\mathcal{Q} = -2\mu\mathcal{Q}. \quad (13)$$

[‡] The exterior derivatives of the coframe 1-forms are given in the Appendix.

2.1. The Papapetrou Field and the Equation $R_{ab}\xi^b = 0$

Using equations (2) and (8), the Papapetrou field in this frame takes the form

$$F = \mathbf{d}(\xi_a \mathbf{d}x^a) = -2 \mathcal{R} \mathbf{d}\mathcal{R} \wedge \mathbf{d}\varphi = -2 \mathbf{i} \mathcal{Q} \mathbf{m} \wedge \overline{\mathbf{m}}. \quad (14)$$

Now, by virtue of the identity $\xi_{b;c;d} - \xi_{b;d;c} = \xi^a R_{abcd}$ and Killing's equations $\xi_{a;b} + \xi_{b;a} = 0$, the Papapetrou field satisfies the equation $F^{ab}{}_{;b} = -2 \xi^{a;b}{}_{;b} = 2 R^a{}_b \xi^b$, so that in vacuum (or when $R^a{}_b \xi^b$ vanishes), the Hodge-dual of F is closed,

$$\mathbf{d}(*F) = \mathbf{d}(-2 \mathcal{Q} \mathbf{l} \wedge \mathbf{n}) = 0. \quad (15)$$

This gives the single equation

$$\delta \mathcal{Q} = 2 \tau \mathcal{Q}. \quad (16)$$

Finally, using equations (13) and (16), the integrability condition $\mathbf{d}\mathbf{d}\mathcal{Q} = 0$ gives

$$D\tau - \delta\rho = \kappa\mu - \rho(\alpha + \beta - \tau), \quad (17)$$

$$D\mu + \Delta\rho = 2(\rho\gamma - \mu\epsilon), \quad (18)$$

$$\Delta\tau + \delta\mu = \nu\rho - \mu(\alpha + \beta + \tau). \quad (19)$$

In summary, we restrict the choice of frame by imposing the five conditions: (i) the components of ξ in the \mathbf{l} , \mathbf{n} , $(\mathbf{m} + \overline{\mathbf{m}})$ directions vanish, and (ii) the gradient of the norm of the KV points in the direction of $\mathbf{m} + \overline{\mathbf{m}}$, which, for a hypersurface-orthogonal KV, implies that the spin coefficients satisfy $\sigma = \rho$, and $\lambda = \mu$. Since $\mathbf{m} + \overline{\mathbf{m}}$ is now hypersurface orthogonal, the condition $\pi + \tau = 0$ follows. The remaining equations obtained in this section are the result of applying the commutators to the functions \mathcal{R} , \mathcal{Q} and requiring $R_{ab}\xi^a\xi^b$ to vanish. Being integrability conditions, they are contained in the Ricci and Bianchi identities as will be shown in the next section.

3. The Reduced NP Equations

We now examine the Ricci identities with $\sigma = \rho$, $\lambda = \mu$, $\pi = -\tau$ and $\overline{\delta} = \delta$ and assuming all spin coefficients real. As these equations give real expressions for Ψ_a , Φ_{ab} we can assume $\Phi_{ba} = \overline{\Phi_{ab}} = \Phi_{ab}$. Then the 18 Ricci equations can be combined to give

- Six equations defining the 5 Ψ_a (the expression for Ψ_2 can be obtained in two ways)

$$\begin{aligned} \Psi_0 &= -2\kappa(\alpha - \beta) + \Phi_{00}, \\ \Psi_1 &= -2\rho(\alpha - \beta) + \Phi_{01}, \\ \Psi_2 &= -2\tau(\alpha - \beta) + \Phi_{02} - 2\Lambda, \\ \Psi_3 &= 2\mu(\alpha - \beta) + \Phi_{12}, \\ \Psi_4 &= 2\nu(\alpha - \beta) + \Phi_{22}. \end{aligned} \quad (20)$$

§ Note that the self-dual part of F , $F - \mathbf{i}^*F$, equals $2\mathbf{i}\mathcal{Q}(\mathbf{l} \wedge \mathbf{n} - \mathbf{m} \wedge \overline{\mathbf{m}})$ so the “canonical frame” defined in this paper based on the properties of the KV coincides with the “preferred frame” (when F is non-null) defined in references [1, 3] based on the coincidence of the null eigendirections of the Papapetrou field with the NP frame vectors \mathbf{l} , \mathbf{n} .

- Five “coframe integrability conditions”, obtained by exterior differentiating equations (81) - (84) in the Appendix. These equations, being independent of the Weyl and Ricci tensor components, are called “eliminant relations” by Chandrasekhar [8]. They are best written using the “divergence operators” \mathcal{D} , \mathcal{A} , \mathcal{S} introduced in [9]:

$$\mathcal{D} \equiv D + \epsilon + \bar{\epsilon} - \rho - \bar{\rho}, \quad \mathcal{A} \equiv \Delta - \gamma - \bar{\gamma} + \mu + \bar{\mu}, \quad \mathcal{S} \equiv \delta + \beta - \bar{\alpha} + \bar{\pi} - \tau. \quad (21)$$

Using these operators, which appear naturally in the exterior differentiation of 3-forms, the coframe integrability conditions read:

$$\mathcal{D}(\tau - 2\beta) - \mathcal{A}\kappa + 2\mathcal{S}\epsilon = 0, \quad (22)$$

$$\mathcal{D}\mu + \mathcal{A}\rho = 0, \quad (23)$$

$$\mathcal{D}\nu + \mathcal{A}(\tau + 2\alpha) - 2\mathcal{S}\gamma = 0, \quad (24)$$

$$\mathcal{D}(\alpha - \beta) = 2\epsilon(\alpha - \beta), \quad (25)$$

$$\mathcal{A}(\alpha - \beta) = -2\gamma(\alpha - \beta). \quad (26)$$

Three of these equations have already been obtained in computing the integrability conditions of $\mathbf{d}\mathcal{R}$, $\mathbf{d}\mathcal{Q}$: Equations (25), (26) are the same as equations (9) and (10), while equation (23) is the same as equation (18).

- Seven remaining equations determining the Ricci tensor. Using the abbreviations

$$G_1 = (\epsilon - \rho)(\tau + \alpha - \beta) - \kappa(\mu + \gamma), \quad (27)$$

$$G_0 = \gamma\rho + \epsilon\mu, \quad (28)$$

$$G_{-1} = (\mu - \gamma)(\tau + \alpha - \beta) - \nu(\rho + \epsilon), \quad (29)$$

$$\Sigma_0 = \tau^2 + \kappa\nu + 2\tau(\alpha - \beta), \quad (30)$$

they take the form

$$\mathcal{D}\rho - \mathcal{S}\kappa = 4\epsilon\rho - 2\kappa(\alpha + \beta) + \Phi_{00}, \quad (31)$$

$$\mathcal{D}(\alpha + \beta) - 2\mathcal{S}\epsilon = 2(G_1 + \Phi_{01}), \quad (32)$$

$$\mathcal{D}\mu - \mathcal{A}\rho + 2\mathcal{S}\tau = 2(-\Sigma_0 + \Phi_{02}), \quad (33)$$

$$\mathcal{D}\gamma - \mathcal{A}\epsilon = -\Sigma_0 - 2G_0 + \Phi_{02} + \Phi_{11} - 3\Lambda, \quad (34)$$

$$\mathcal{S}(\alpha - \beta) = -\Phi_{02} + \Phi_{11} + 3\Lambda, \quad (35)$$

$$2\mathcal{S}\gamma - \mathcal{A}(\alpha + \beta) = 2(G_{-1} + \Phi_{12}), \quad (36)$$

$$\mathcal{S}\nu - \mathcal{A}\mu = 4\gamma\mu - 2\nu(\alpha + \beta) + \Phi_{22}. \quad (37)$$

Using the definition of \mathcal{Q} (equation (12)) and (7), we find that equation (35) is the same as (16) when

$$\Phi_{11} + 3\Lambda - \Phi_{02} = \frac{1}{2}R_{ab}\xi^a\xi^b = 0. \quad (38)$$

3.1. The Bianchi Identities

When the values of Ψ_a given by (20) together with $\sigma = \rho$, $\lambda = \mu$, $\pi = -\tau$ and $\bar{\delta} = \delta$ are substituted into the Bianchi identities, and the Ricci identities with $\Phi_{ba} = \Phi_{ab}$ are

used, one finds, after a long calculation, that they can be reduced to five independent equations: the three contracted Bianchi identities, involving the components of the Ricci tensor only,

$$D(\Phi_{11} + 3\Lambda) - 2\delta\Phi_{01} + \Delta\Phi_{00} = 2(2\gamma - \mu)\Phi_{00} - 2(2\alpha + 3\tau)\Phi_{01} + 2\rho(\Phi_{02} + 2\Phi_{11}) - 2\kappa\Phi_{12}, \quad (39)$$

$$D\Phi_{12} - \delta(\Phi_{02} + \Phi_{11} - 3\Lambda) + \Delta\Phi_{01} = \nu\Phi_{00} + 2(\gamma - 2\mu)\Phi_{01} - 2(\alpha - \beta + \tau)\Phi_{02} - 4\tau\Phi_{11} + 2(2\rho - \epsilon)\Phi_{12} - \kappa\Phi_{22}, \quad (40)$$

$$D\Phi_{22} - 2\delta\Phi_{12} + \Delta(\Phi_{11} + 3\Lambda) = 2\nu\Phi_{01} - 2\mu(\Phi_{02} + 2\Phi_{11}) + 2(2\beta - 3\tau)\Phi_{12} + 2(\rho - 2\epsilon)\Phi_{22}, \quad (41)$$

and the following two equations

$$(D - 4\rho)(\Phi_{11} + 3\Lambda - \Phi_{02}) + 2(\alpha - \beta)(D\tau - \delta\rho + \rho(\alpha + \beta - \tau) - \kappa\mu) = 0, \quad (42)$$

$$(\Delta + 4\mu)(\Phi_{11} + 3\Lambda - \Phi_{02}) + 2(\alpha - \beta)(\Delta\tau + \delta\mu + \mu(\alpha + \beta + \tau) - \nu\rho) = 0. \quad (43)$$

These last two equations reduce to equations (17), (19) when (38) holds. This was to be expected since equations (17)-(19) were derived using (16), which is equation (35) with vanishing rhs. Collecting everything together we conclude that, in this frame, the complete set of NP equations for spacetimes with one hypersurface-orthogonal KV, are given by the 17 equations in this section (12 Ricci and 5 Bianchi). The Weyl scalars are given by (20) and the Ricci tensor identically satisfies $\xi_{[a}R_{b]c}\xi^c = 0$ ($\leftrightarrow \Phi_{ba} = \Phi_{ab}$).

In section 4, we show that two of these equations (25, 26) can be satisfied identically by using equation (13) to *define* $\rho\mathbf{n} - \mu\mathbf{l}$. When the Ricci tensor satisfies $R_{ab}\xi^a\xi^b = 0$, one more equation (35) can be reduced to an identity by choosing certain metric functions appropriately. Finally, in vacuum, the three contracted Bianchi identities disappear, so we are left with 11 (real) equations.

3.2. Main and Subsidiary Equations

Ever since the classic paper of Bondi *et al.* [10], it is instructive to split the seven Ricci equations (31-37) into 4 “main” and 3 “subsidiary” equations, such that, when the “main” equations are satisfied everywhere, the contracted Bianchi identities impose extra conditions on an “initial” hypersurface. In a paper that has received less attention than it deserves[¶], Waylen [5] has studied the axisymmetric vacuum Einstein equations. Using a particular coordinate system (one coordinate being the scalar function \mathcal{R}) in which (38) is satisfied identically, he has shown that

[¶] Apart from three equations obtained by letting the commutators (85) act on a function independent of \mathcal{R} , \mathcal{Q} .

[¶] This paper gives a prescription for constructing a power series solution of the axisymmetric vacuum Einstein equations, valid near the axis, and depending on an arbitrary function of two variables – much as [10] gives the series solution near future null infinity in terms of an arbitrary “news” function $c(u, \theta)$. However, unlike [10] where four main equations need to be satisfied, in [5] one main equation is satisfied identically.

- (i) Near the symmetry axis, analytic (power series) solutions depending on an arbitrary function of two variables can be constructed by solving the “main” equations.
- (ii) Two of the Bianchi identities can then be satisfied identically by letting the three remaining components of the Ricci tensor be proportional to the derivatives of a single scalar function \mathcal{P} .
- (iii) The third Bianchi identity then implies that the function \mathcal{P} satisfies the wave equation; and the only solution of that equation that behaves well on the axis is the trivial solution $\mathcal{P} = 0$.

Remarkably, Waylen’s integration of the Bianchi identities can be carried out in our formalism *without resorting to a particular coordinate system*. Following Waylen, let us take as main equations the set $\{\Phi_{00} = 0, \Phi_{11} = 3\Lambda, \Phi_{22} = 0\}$ together with equation (38). Then, expressing the non-vanishing Ricci components in terms of $\Phi_{01}, \Phi_{11}, \Phi_{12}$, the Bianchi identities become

$$D\Phi_{11} - \delta\Phi_{01} + (2\alpha + 3\tau)\Phi_{01} - 4\rho\Phi_{11} + \kappa\Phi_{12} = 0, \quad (44)$$

$$D\Phi_{12} - 2\delta\Phi_{11} + \Delta\Phi_{01} - 2(\gamma - 2\mu)\Phi_{01} + 4(\alpha - \beta + 2\tau)\Phi_{11} + 2(\epsilon - 2\rho)\Phi_{12} = 0, \quad (45)$$

$$\delta\Phi_{12} - \Delta\Phi_{11} + \nu\Phi_{01} - 4\mu\Phi_{11} + (2\beta - 3\tau)\Phi_{12} = 0. \quad (46)$$

Now, the first and last of these equations are satisfied identically, by virtue of the commutators (85) and the known derivatives of \mathcal{Q} and \mathcal{R} , if we set

$$\Phi_{01} = \frac{\mathcal{Q}}{\mathcal{R}} D\mathcal{P}, \quad \Phi_{11} = \frac{\mathcal{Q}}{\mathcal{R}} \delta\mathcal{P}, \quad \Phi_{12} = \frac{\mathcal{Q}}{\mathcal{R}} \Delta\mathcal{P}, \quad (47)$$

for some scalar function \mathcal{P} . The second equation (apart from an overall factor \mathcal{Q}/\mathcal{R}) then becomes

$$\Delta(D\mathcal{P}) + \not{D}(\Delta\mathcal{P}) - 2\delta(\delta\mathcal{P}) = 0, \quad (48)$$

which is the wave equation for \mathcal{P} , the left-hand-side being equal to $-*\mathbf{d}(*\mathbf{d}\mathcal{P})$.

The “main” equations to be solved are then

$$\not{D}\rho - \delta\kappa = 4\epsilon\rho - 2\kappa(\alpha + \beta), \quad (49)$$

$$\not{D}(\mu - 2\gamma) - \Delta(\rho - 2\epsilon) + 2\delta\tau = 4G_0, \quad (50)$$

$$\delta\nu - \Delta\mu = 4\gamma\mu - 2\nu(\alpha + \beta), \quad (51)$$

together with the “integrability conditions” – equations (17) - (19) and (22) - (26).⁺ And Waylen’s result suggests that a solution of these equations that is well-behaved near the axis $\mathcal{R} = 0$ will also satisfy the remaining three Ricci equations.

4. General Coordinate Expressions

Let x^a ($a = 1, 2, 3$) be the coordinate labels of an arbitrary coordinate system on the 3-surfaces orthogonal to the Killing trajectories. Then the properly normalized complex

⁺ These integrability conditions are identities when the spin coefficients are expressed in terms of the derivatives of the components of the frame vectors in a coordinate frame.

null (co-) vector, $(\mathbf{m})\delta$ will have the form (see (8) and the definition of \mathcal{Q} , (12))

$$\mathbf{m} = \frac{-1}{\sqrt{2}} \left(\frac{\mathcal{R}_{,a} \mathbf{d}x^a}{\mathcal{Q}} + \mathbf{i} \mathcal{R} \mathbf{d}\varphi \right), \quad \delta = \frac{1}{\sqrt{2}} \left(\frac{\mathcal{Q} H^a}{H^c \mathcal{R}_{,c}} \frac{\partial}{\partial x^a} + \frac{\mathbf{i}}{\mathcal{R}} \frac{\partial}{\partial \varphi} \right), \quad (52)$$

for some vector H^a , defined up to an overall scale factor. Now, by equations (13) and (16), $\mathbf{d}\mathcal{Q} = 2\mathcal{Q}(\rho \mathbf{n} - \mu \mathbf{l} - \tau(\mathbf{m} + \overline{\mathbf{m}}))$, so that the co-vector $\rho \mathbf{n} - \mu \mathbf{l}$, using (52), can be written

$$\rho \mathbf{n} - \mu \mathbf{l} = \frac{1}{2\mathcal{Q}} \left(\mathcal{Q}_{,a} - \frac{H^s \mathcal{Q}_{,s}}{H^t \mathcal{R}_{,t}} \mathcal{R}_{,a} \right) \mathbf{d}x^a. \quad (53)$$

Next we observe that the vector $\rho \Delta + \mu D$ gives zero (by equations (5) and (13)) when acting on the scalars \mathcal{R} , \mathcal{Q} . The unique such vector in a 3-dimensional space is

$$\rho \Delta + \mu D = \frac{\epsilon^{abc} \mathcal{R}_{,a} \mathcal{Q}_{,b}}{2K} \frac{\partial}{\partial x^c}, \quad (54)$$

where K is a proportionality factor. Then the properly normalized co-vector $\rho \mathbf{n} + \mu \mathbf{l}$ will have the form

$$\rho \mathbf{n} + \mu \mathbf{l} = \frac{4\rho\mu K}{\mathbf{T} \cdot (\nabla \mathcal{R} \times \nabla \mathcal{Q})} \left(T_a - \frac{H^s T_s}{H^t \mathcal{R}_{,t}} \mathcal{R}_{,a} \right) \mathbf{d}x^a, \quad (55)$$

where the T_a are three arbitrary functions and

$$\mathbf{T} \cdot (\nabla \mathcal{R} \times \nabla \mathcal{Q}) = \epsilon^{abc} T_a \mathcal{R}_{,b} \mathcal{Q}_{,c}. \quad (56)$$

Observe that in (55) the T_a are defined up to an overall scale factor and up to the addition of a multiple of $\mathcal{R}_{,a}$. Finally, the vector $\rho \Delta - \mu D$, which is orthogonal to both $\mathbf{m} + \overline{\mathbf{m}}$ and $\rho \mathbf{n} + \mu \mathbf{l}$ and gives $-4\mu\rho\mathcal{Q}$ when acting on \mathcal{Q} , is given by

$$\rho \Delta - \mu D = 4\mu\rho\mathcal{Q} \frac{\epsilon^{abc} \mathcal{R}_{,a} T_b}{\mathbf{T} \cdot (\nabla \mathcal{R} \times \nabla \mathcal{Q})} \frac{\partial}{\partial x^c}. \quad (57)$$

By equations (53) and (55), the tensor product of \mathbf{l} , \mathbf{n} depends on the product of the functions μ and ρ only, so the functions entering the metric are \mathcal{R} , \mathcal{Q} , H^a , K , $\mu\rho$, T_a – a total of 10 functions. Remembering that the vectors H^a , T_a are defined up to scale and that the expressions are invariant if a multiple of $\mathcal{R}_{,a}$ is added to T_a , we conclude that, in an arbitrary coordinate system, the metric depends on 7 independent functions, as expected.

This three-parameter freedom in choosing the metric functions H^a , T_a can be used to simplify the equations. For example, using equations (53) and (55), the closed 2-form $\mathcal{Q}\mathbf{l} \wedge \mathbf{n}$ is given by

$$\mathcal{Q}\mathbf{l} \wedge \mathbf{n} = -\frac{K}{H^s \mathcal{R}_{,s}} \epsilon_{abc} H^a \mathbf{d}x^b \wedge \mathbf{d}x^c. \quad (58)$$

We can now fix the scale of H by imposing the condition $H^s \mathcal{R}_{,s} = K$. Equation (15) then becomes $H^a_{,a} = 0$ – an equation that can be integrated in terms of a vector potential.

Collecting everything together, and introducing the index-free notation,

$$\nabla_{\mathbf{H}} \mathcal{R} = H^a \mathcal{R}_{,a}, \quad \mathbf{H} \cdot \mathbf{T} = H^a T_a, \quad (59)$$

we conclude that, in an arbitrary coordinate basis, the (co-) frame vectors have the form

$$\rho \mathbf{n} + \mu \mathbf{l} = 4\mu\rho \frac{(\nabla_{\mathbf{H}}\mathcal{R} T_a - \mathbf{H} \cdot \mathbf{T} \mathcal{R}_{,a})\mathbf{d}x^a}{\mathbf{T} \cdot (\nabla\mathcal{R} \times \nabla\mathcal{Q})}, \quad (60)$$

$$\rho \mathbf{n} - \mu \mathbf{l} = \frac{1}{2\mathcal{Q}}(\mathcal{Q}_{,a} - \frac{\nabla_{\mathbf{H}}\mathcal{Q}}{\nabla_{\mathbf{H}}\mathcal{R}}\mathcal{R}_{,a})\mathbf{d}x^a, \quad (61)$$

$$\mathbf{m} = \frac{-1}{\sqrt{2}} \left(\frac{\mathcal{R}_{,a}\mathbf{d}x^a}{\mathcal{Q}} + i\mathcal{R} \mathbf{d}\varphi \right), \quad (62)$$

$$\rho \Delta + \mu D = \frac{\epsilon^{abc} \mathcal{R}_{,a} \mathcal{Q}_{,b}}{2 \nabla_{\mathbf{H}}\mathcal{R}} \frac{\partial}{\partial x^c}, \quad (63)$$

$$\rho \Delta - \mu D = 4\mu\rho \mathcal{Q} \frac{\epsilon^{abc} \mathcal{R}_{,a} T_b}{\mathbf{T} \cdot (\nabla\mathcal{R} \times \nabla\mathcal{Q})} \frac{\partial}{\partial x^c}, \quad (64)$$

$$\delta = \frac{1}{\sqrt{2}} \left(\frac{\mathcal{Q}}{\nabla_{\mathbf{H}}\mathcal{R}} H^a \frac{\partial}{\partial x^a} + \frac{i}{\mathcal{R}} \frac{\partial}{\partial \varphi} \right). \quad (65)$$

These expressions are invariant under a redefinition of the T_a according to

$$T_a \rightarrow T'_a = \lambda_1 T_a + \lambda_2 \mathcal{R}_{,a}, \quad \lambda_1, \lambda_2 \text{ arbitrary}, \quad (66)$$

while equation (38) is satisfied by choosing

$$H^a = \epsilon^{abc} \mathcal{A}_{b,c} \quad \text{for some vector potential } \mathcal{A}_a. \quad (67)$$

Finally, we observe that the linear combinations of the vectors \mathbf{l}, \mathbf{n} appearing in equations (60-64) are invariant under the boost freedom (6).

5. Possible Choices of Frame and Coordinate Degrees of Freedom

The expressions for the NP frame vectors given in the previous section are invariant under two distinct types of transformation:

- (i) the remaining freedom in the choice of unknown functions (the ratio μ/ρ and the arbitrariness in the definition of T_a), and
- (ii) an arbitrary change in coordinates $x^a \rightarrow x'^a(x^b)$, that can be eliminated by imposing three “coordinate conditions” on the 7 independent metric functions.

Ideally, the freedom in the first type of transformation should be used to bring the remaining three “main” equations to as simple a form as possible, as was done with (35) and the scaling of H^a . Then the second type of transformation can be used to seek solutions of these equations in a system of coordinates that is adapted to the specific physical system one is interested in. In that way, it will be easier to relate the arbitrary functions of integration entering the solution (and the three “coordinate conditions”) to properties of the physical system. That the choice of coordinates is important in specifying a particular problem is evident from the following consideration: the “general” (depending on an arbitrary function of two variables) approximate solutions obtained in references [5] and [10], using geometrically defined coordinates, offer no clues as to how the arbitrary functions in these solutions are to be chosen so as to give the metric

describing, say, two coalescing Schwarzschild black holes rather than that of any other distribution of matter along the axis. For the two-black-hole problem, the functions of integration would be expected to depend ultimately on the properties of the two world lines and the two masses – the physical data needed to specify the problem. These are functions of one variable only – a parameter (proper time) along each world line. With this in mind, we have developed and studied a coordinate system that uses these two parameters as coordinates and is thus appropriate for the two-black-hole problem. It will be presented in a future publication. For the rest of this section we will discuss possible choices of the first kind of transformation.

The freedom in the choice of T_a can be used to make the scalars $\mathbf{T} \cdot (\nabla \mathcal{R} \times \nabla \mathcal{Q})$ and $\mathbf{H} \cdot \mathbf{T}$ equal to anything one pleases. And it is not unreasonable to expect that one can exploit this freedom to bring one (or more) combination(s) of the equations to a form that can be solved. The fact, however, that these scalars involve almost all of the free functions in a non-trivial way makes this task extremely difficult.

Turning now to the boost freedom (6), several choices suggest themselves: the simplest is to use this freedom to set the ratio μ/ρ equal to unity. (In fact, Ludwig [3] uses this choice in some cases). However, the main equations do not become any simpler* with this choice. Another choice is to use the boost freedom (6), under which $\alpha + \beta \rightarrow \alpha + \beta + \delta(\ln A)$, to make the sum $\alpha + \beta$ vanish. But this again does not lead to equations that we know how to handle.

A more promising possibility arises in considering the main equation (50). The effect of a boost on G_0 , defined in (28), is

$$G_0 \rightarrow G_0 + \frac{1}{2}(\rho \Delta + \mu D) \ln A, \quad (68)$$

so, in principle, A can be chosen to make G_0 vanish. Then equation (50) becomes a total divergence, which again can be “integrated” in terms of a vector potential. Specifically, when $\rho \gamma + \mu \epsilon = 0$, equation (50) implies that the following 2-form is closed:

$$F_2 = \mathcal{R} [(\mu - 2\gamma) \mathbf{l} \wedge (\mathbf{m} + \overline{\mathbf{m}}) + (\rho - 2\epsilon) \mathbf{n} \wedge (\mathbf{m} + \overline{\mathbf{m}}) + 2\tau \mathbf{l} \wedge \mathbf{n}]. \quad (69)$$

However, we have not been able to obtain manageable expressions using the vector potential arising from $\mathbf{d}F_2 = 0$ together with $\rho \gamma + \mu \epsilon = 0$.

Finally, the observation that, under the boost (6) the connection one-form

$$\omega_{01} = \gamma \mathbf{l} + \epsilon \mathbf{n} - \alpha \mathbf{m} - \beta \overline{\mathbf{m}} \quad (70)$$

transforms according to $\omega_{01} \rightarrow \omega_{01} + \mathbf{d}A/A$, suggests that we can impose a Lorentz-type gauge condition on ω_{01}

$$-^* \mathbf{d}(^* \omega_{01}) = \not{D}\gamma + \not{D}\epsilon - \not{\delta}(\alpha + \beta) = 0, \quad (71)$$

which implies that the 2-form

$$F_3 = \mathcal{R} [\gamma \mathbf{l} \wedge (\mathbf{m} + \overline{\mathbf{m}}) - \epsilon \mathbf{n} \wedge (\mathbf{m} + \overline{\mathbf{m}}) - (\alpha + \beta) \mathbf{l} \wedge \mathbf{n}] \quad (72)$$

is closed. There are, clearly, many other possibilities.

* Simplicity is, of course, a subjective criterion. What we are really looking for are equations that we know how to integrate!

5.1. Properties of Gravitational Principal Null Directions

Instead of trying to simplify the general equations as much as possible, one can seek solutions that satisfy additional conditions. Such conditions can be suggested by examining the properties of the Gravitational Principal Null Directions (GPNDs) of the spacetime. For example, equations (20) imply that, in vacuum, the frame vector \mathbf{l} is a GPND ($\Psi_0 = 0$) if and only if it is geodesic ($\kappa = 0$), and the same is true of \mathbf{n} . So $\kappa = 0 = \nu$ are obvious conditions to use in the search for special solutions.

Consider now an arbitrary null vector in the 3-space orthogonal to the Killing trajectories $\mathbf{l} + z^2 \mathbf{n} + z(\mathbf{m} + \overline{\mathbf{m}})$, parametrized by some real parameter z . The condition that it is a GPND is $\Psi_0 + 4\Psi_1 z + 6\Psi_2 z^2 + 4\Psi_3 z^3 + \Psi_4 z^4 = 0$, which, using (20) in vacuum becomes

$$\kappa + 6\tau z^2 - \nu z^4 + 4(\rho z - \mu z^3) = 0. \quad (73)$$

We now observe that, if we choose $z = \pm \sqrt{\rho/\mu}$ (assuming that ρ and μ have the same sign), and require that the spin coefficients satisfy the single relation

$$\kappa \mu^2 - \nu \rho^2 + 6\mu \rho \tau = 0, \quad (74)$$

then *both* null vectors

$$N_{\pm} = \mu \mathbf{l} + \rho \mathbf{n} \pm \sqrt{\mu \rho}(\mathbf{m} + \overline{\mathbf{m}}) \quad (75)$$

will be GPNDs. Equation (74) is invariant under the boost transformation (6). Nevertheless, it can be considered as determining the ratio ρ/μ in terms of κ, ν, τ . Then, provided that (74) gives two distinct real and positive solutions for this ratio, equation (75) will determine altogether four distinct GPNDs. These four GPNDs and the frame vectors $\mathbf{l}, \mathbf{n}, \mathbf{m}, \overline{\mathbf{m}}$ will then be related in a way that is analogous to the way the four GPNDs of a Petrov-type I spacetime are related to the geometrically determined “Weyl canonical frame” defined in [11]. Thus, condition (74) on the spin coefficients can be interpreted as the condition that the “canonical frame” defined in this paper coincides with the “Weyl canonical frame” of [11]. Alternatively, one can impose the condition that one of the null vectors (75) is hypersurface orthogonal, say $N_+ \wedge \mathbf{d}N_+ = 0$, thus defining a family of Bondi-like null hypersurfaces.

Note that the extra conditions proposed here do not necessarily imply that the Weyl tensor is algebraically special, even though further analysis of the equations may lead to that conclusion.

We intend to consider further some of the possibilities suggested here in future publications.

Note: The results presented in this paper involve extensive calculations. All equations given have been checked using the computer algebra program MATHEMATICA together with the author’s package “*Exterior Differential Calculus*”, which is available at www.inp.demokritos.gr/~sbonano/EDC/.

Appendix

The exterior derivatives of the basis co-vectors are given by (see [6], §4.13)

$$\mathbf{d}l = (\alpha + \bar{\beta} - \bar{\tau})l \wedge \mathbf{m} + (\bar{\alpha} + \beta - \tau)l \wedge \bar{\mathbf{m}} - (\epsilon + \bar{\epsilon})l \wedge \mathbf{n} \quad (76)$$

$$+ (\rho - \bar{\rho})\mathbf{m} \wedge \bar{\mathbf{m}} + \bar{\kappa}\mathbf{m} \wedge \mathbf{n} + \kappa\bar{\mathbf{m}} \wedge \mathbf{n},$$

$$\mathbf{d}m = (\gamma - \bar{\gamma} + \bar{\mu})l \wedge \mathbf{m} + \bar{\lambda}l \wedge \bar{\mathbf{m}} - (\bar{\pi} + \tau)l \wedge \mathbf{n} \quad (77)$$

$$+ (-\bar{\alpha} + \beta)\mathbf{m} \wedge \bar{\mathbf{m}} + (-\epsilon + \bar{\epsilon} + \rho)\mathbf{m} \wedge \mathbf{n} + \sigma\bar{\mathbf{m}} \wedge \mathbf{n},$$

$$\mathbf{d}n = \nu l \wedge \mathbf{m} + \bar{\nu}l \wedge \bar{\mathbf{m}} - (\gamma + \bar{\gamma})l \wedge \mathbf{n} + (\mu - \bar{\mu})\mathbf{m} \wedge \bar{\mathbf{m}} \quad (78)$$

$$+ (\alpha + \bar{\beta} - \pi)\mathbf{m} \wedge \mathbf{n} + (\bar{\alpha} + \beta - \bar{\pi})\bar{\mathbf{m}} \wedge \mathbf{n}.$$

The assumption that the Killing 1-form is hypersurface orthogonal and in the direction $\mathbf{m} - \bar{\mathbf{m}}$ implies that $(\mathbf{m} - \bar{\mathbf{m}}) \wedge \mathbf{d}(\mathbf{m} - \bar{\mathbf{m}}) = 0$, which gives the equations

$$\mu - 2\gamma + \lambda = \bar{\mu} - 2\bar{\gamma} + \bar{\lambda}, \quad \rho - 2\epsilon + \sigma = \bar{\rho} - 2\bar{\epsilon} + \bar{\sigma}, \quad \tau - \pi = \bar{\tau} - \bar{\pi}. \quad (79)$$

In addition, it requires that the exterior derivatives of $l, n, (\mathbf{m} + \bar{\mathbf{m}})$ must be expressible as linear combinations of $l \wedge n, l \wedge (\mathbf{m} + \bar{\mathbf{m}}), n \wedge (\mathbf{m} + \bar{\mathbf{m}})$ only. Requiring that the coefficients of $\mathbf{m} \wedge \bar{\mathbf{m}}, l \wedge (\mathbf{m} - \bar{\mathbf{m}}), n \wedge (\mathbf{m} - \bar{\mathbf{m}})$ vanish, we obtain

$$\begin{aligned} \rho &= \bar{\rho}, & \kappa &= \bar{\kappa}, & \tau + \alpha - \beta &= \bar{\tau} + \bar{\alpha} - \bar{\beta}, \\ \alpha + \beta &= \bar{\alpha} + \bar{\beta}, & \rho - 2\epsilon - \sigma &= \bar{\rho} - 2\bar{\epsilon} - \bar{\sigma}, & \mu - 2\gamma - \lambda &= \bar{\mu} - 2\bar{\gamma} - \bar{\lambda}, \\ \mu &= \bar{\mu}, & \nu &= \bar{\nu}, & \pi - \alpha + \beta &= \bar{\pi} - \bar{\alpha} + \bar{\beta}. \end{aligned} \quad (80)$$

The unique solution of the 12 linear equations (79), (80) is that all 12 spin coefficients are real.

With all spin coefficients real and the further choice of frame made in section 2, for which $\lambda = \mu, \sigma = \rho, \pi = -\tau$, the exterior derivatives of $l, n, (\mathbf{m} + \bar{\mathbf{m}}), (\mathbf{m} - \bar{\mathbf{m}})$ are given by

$$\mathbf{d}l = (\alpha + \beta - \tau)l \wedge (\mathbf{m} + \bar{\mathbf{m}}) - 2\epsilon l \wedge \mathbf{n} + \kappa(\mathbf{m} + \bar{\mathbf{m}}) \wedge \mathbf{n}, \quad (81)$$

$$\mathbf{d}n = \nu l \wedge (\mathbf{m} + \bar{\mathbf{m}}) - 2\gamma l \wedge \mathbf{n} + (\alpha + \beta + \tau)(\mathbf{m} + \bar{\mathbf{m}}) \wedge \mathbf{n}, \quad (82)$$

$$\mathbf{d}(\mathbf{m} + \bar{\mathbf{m}}) = 2\mu l \wedge (\mathbf{m} + \bar{\mathbf{m}}) + 2\rho(\mathbf{m} + \bar{\mathbf{m}}) \wedge \mathbf{n}, \quad (83)$$

$$\mathbf{d}(\mathbf{m} - \bar{\mathbf{m}}) = (\alpha - \beta)(\mathbf{m} + \bar{\mathbf{m}}) \wedge (\mathbf{m} - \bar{\mathbf{m}}). \quad (84)$$

These equations are equivalent to the commutators and define the spin coefficients when the frame vectors are given in a coordinate basis. They are used repeatedly in exterior differentiation of forms.

Acting on functions that do not depend on φ , the three non-trivial commutators are:

$$\begin{aligned} [\delta, D] &= (\alpha + \beta + \tau)D - 2\rho\delta + \kappa\Delta, \\ [\Delta, D] &= 2\gamma D + 2\epsilon\Delta, \\ [\Delta, \delta] &= \nu D - 2\mu\delta + (\alpha + \beta - \tau)\Delta. \end{aligned} \quad (85)$$

The equations resulting from the action of these commutators on the scalars \mathcal{R}, \mathcal{Q} are contained in the Ricci equations obtained in section 3.

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